Learning objectives: should understand the necessary conditions for these waves to exist, wave propagation & energy dispersion

1. Mid-latitude Rossby waves ($\beta$ effect) - continue;
2. Effects of side boundaries: coastally-trapped waves.
(i) For high frequency waves

\[ \omega^2 = k^2 c^2 + f_0^2. \]

(ii) For low frequency waves with small \( \omega \),

Long:

\[ \omega = -\frac{\beta c^2}{f_0^2} k = -c_r k. \]

Short:

\[ \omega = -\frac{\beta}{k}. \]

\[ c_g = \frac{\partial \omega}{\partial k} = \frac{\beta}{k^2}, \]

\[ c_p = \frac{\omega}{k} = -\frac{\beta}{k^2}. \]
Previous class, for \( f = f_0 + \beta y \)

Dispersion relation:

\[
(k + \frac{\beta}{2\omega})^2 + l^2 = \frac{\omega^2 - f_0^2}{c^2} + \frac{\beta^2}{4\omega^2}.
\]

where \( c = \pm \sqrt{gH} \)

To simplify the case, let \( l = 0 \) and look at 1-dimensional situation. We have

\[
(k + \frac{\beta}{2\omega})^2 = \frac{\omega^2 - f_0^2}{c^2} + \frac{\beta^2}{4\omega^2}.
\]
(i) For high frequency waves with large $\omega$, the dispersion relation can be approximated by:

$$\omega^2 = \kappa^2 c^2 + f^2,$$

This is long gravity wave under the influence of $f$. It is also called inertial gravity wave.

They are not influenced much by beta-variation of $f$!
(ii) For low frequency waves with small $\omega$,

$$k = -\frac{\beta}{2\omega}[1 \mp (1 - 2\frac{f_0^2 \omega^2}{\beta^2 c^2})]$$

These waves do not exist in $f=$constant case; their existence is due to the introduction of

They are called Rossby (or planetary) waves.
Choose ‘-’ sign:

\[ k = -\frac{\beta}{2\omega}(1 - 1 + 2 \frac{f_0^2 \omega^2}{\beta^2 c^2}) = -\frac{\omega f_0^2}{\beta c^2}, \]

or, \[ \omega = -\frac{\beta c^2}{f_0^2} k = -c_r k. \]

Since \( c_g = \frac{\partial \omega}{\partial k} = -c_r \) is independent of frequency and wavenumber, they are non-dispersive.

\[ c_g = c_p = -\frac{\beta c^2}{f_0^2} \quad \text{They are long Rossby waves and propagate westward; speed decreases as latitude increases.} \]

Here, \( c \) is vertical mode speed: either a baroclinic or the barotropic mode speed.
Choose the ‘+’ sign,

\[ \omega = -\frac{\beta}{k}. \]

\[ c_g = \frac{\partial \omega}{\partial k} = \frac{\beta}{k^2}, \quad c_p = \frac{\omega}{k} = -\frac{\beta}{k^2} \]

They are short Rossby waves. Group velocity propagates eastward but phase propagates westward. They are dispersive.
Dispersion curves for free waves in mid-latitude plane:

Short Rossby waves are hardly seen in the ocean interior because (1) they are too short to be effectively excited by large-scale winds, (2) mixing in the ocean acts strongly on short and slow waves.
Rossby waves in mid-latitude $\beta-$ plane

(i) Existence of Rossby waves: $\beta$, the variation of Planetary vorticity ($f$) with latitude ($\phi$),

$$\beta = \frac{\partial f}{\partial y}.$$

(ii) They are low frequency waves.
Oceanic adjustment

a) Initial perturbation

b) Gravity wave radiation

c) Geostrophy
If there is persistent wind forcing: the equilibrium state is "Sverdrup-balance". Will be introduced later.
Annual period Rossby Wave in eastern tropical Pacific

Comparison of annual cycle anomalies of observed 20 °C depth (left panels) and the Rossby wave model solution (Section 4.2.1) (right panels), for four average seasons (indicated to the left of each row). The common color key is at right, with contour interval of 5 m. Positive values (red) indicate deep anomalies and negative values (blue) indicate shallow anomalies.

Kessler 1990. JGR
Observed mid-latitude Rossby waves by TOPEX satellite altimetry

Mid-latitude Pacific

As you can see, they propagate westward, with a decreasing speed with increasing latitude
2. Effects of side boundaries: coastally-trapped waves

Coasts act as waveguides.
a) Coastal kelvin wave.

Coast. Vertical walls.
For simplicity, we again use the shallow water equation with constant $f$: $f = f_0 > 0$ (Northern Hemisphere)

Find solutions for $v=0$, subject to boundary condition: $v = 0, \, \partial y = 0$.

$$\frac{\partial u}{\partial t} = -g \frac{\partial \eta}{\partial x},$$

$$fu = -g \frac{\partial \eta}{\partial y},$$

$$\frac{\partial \eta}{\partial t} + H \frac{\partial u}{\partial x} = 0.$$
From the above equations, we can obtain,

\[ u_{tt} = gH u_{xx} \]

Assume wave form,

\[ u = u(y) e^{i(kx - \omega t)} \]

\[ \omega = \pm kc, \]

where \[ c = \sqrt{gH} \]

This is just like the dispersion relation for long, surface gravity waves in non-rotating system.

*The existence of f does not affect the dispersion relation!* What’s the effect of f, then?
\[ \eta = \eta(y)e^{i(kx-\omega t)}, \quad u = u(y)e^{i(kx-\omega t)}, \]

The set of equations yields,
\[
\eta_y = -\frac{f}{\omega} \eta, \quad \text{and choose } \omega = -kc,
\]
\[
\eta_y = +\frac{f}{c} \eta,
\]
Thus,
\[
\eta = \eta_0 e^{+\frac{f}{c}y} e^{i(kx-\omega t)}.
\]

*Since sea level increases with the increase of \( y \), which is farther and farther away from the coast, it is not a reasonable solution because energy should decay away from the energy source; for this case the source is the coast.*
Choosing $\omega = kc$, we obtain:

$$\eta = \eta_0 e^{-\frac{f}{c}y} e^{i(kx-\omega t)},$$

and this solution obtains a maximum at the coast and decays away from it. Reasonable.

**Coastal Kelvin waves:** propagate with the coast to its right (left) in Northern (Southern) Hemisphere.

Solutions are trapped to the coast, decaying away from it exponentially, with an e-folding scale of $\frac{c}{f}$, the Rossby radius of deformation.
Satellite Observed Sea Surface Height anomalies: coastal Kelvin waves in The Bay of Bengal of the Indian Ocean
(Rao et al. 2010, DSR)

(Multi-year mean)
b) Continental shelf waves

\[ y \]

- deep
- shallow

Coast
\[
\frac{\partial u}{\partial t} - fv = -g \frac{\partial \eta}{\partial x}, \\
\frac{\partial v}{\partial t} + fu = -g \frac{\partial \eta}{\partial y}, \\
\frac{\partial \eta}{\partial t} + \frac{\partial (Hu)}{\partial x} + \frac{\partial ( Hv)}{\partial y} = 0.
\]

Analytic solution, more complicated.

\[\frac{\partial H}{\partial y}\] acts as \[\beta = \frac{\partial f}{\partial y}\]

Topographic Rossby waves
Propagation: like coastal Kelvin waves;

Shelf waves: dispersive.

Mechanism: like Rossby waves: Potential vorticity conservation:

\[ PV = \frac{\zeta + f}{H} = \text{constant} \]

Near the coast, scale is small, \( f \approx \text{constant} \)
\[ \frac{f}{H_u} = \frac{\zeta + f}{H_n} \]

**NH, south coast**

- **Deep**
  - T0
  - T1

- **Shallow**

- **Positive relative vorticity**
- **Negative relative vorticity**
NH, North Coast

Rossby waves

Topographic waves

North $\zeta < 0$

South $\zeta > 0$

Shallow

Deep

Waveform motion

$\frac{D}{Dt} \left( f + \frac{\zeta}{h} \right) = 0$